International Conference on Next Generation Teletraffic and WiredWireless Advanced Networking
NEW2AN, St. Petersburg, Russia. February 2–7, 2004.

The Derivation of a Stable Recursion for Multi–Service Models

Iversen, V.B.* Stepanov, S.N.† Kostrov, V.O.†

Abstract

In this paper we describe an approach that can be used for constructing numerically stable versions of recursive algorithms for estimation of performance measures of multi-rate models that appear in description of common sharing of bandwidth by a number of connections. The approach is a generalisation for multi-rate models of the well known recurrence formula for Erlang's B–formula. At each step of recurrence we deal with normalised values of state probabilities used for estimation of main stationary performance measures. This makes it very well suited for solving problems of dimensioning. In addition to being numerically stable, the main positive features of the suggested approach are its simplicity and small destorage and computational requirements.

1 Introduction

Let us consider a single link traffic model, where the link transmission capacity is represented by k basic bandwidth units [1]. In the model we have one Poissonian flow of demands for connections with intensity λ which we for simplicity will name by calls. We suppose that each call uses one bandwidth unit for the time of connection having mean value equal to one. To simplify the description of the model dynamics we suppose that the duration of connection (service time) is exponentially distributed with parameter equal to one, but most of the obtained results are valid for general service time distributions.

Let $i(t)$ be number of calls being served at time t. The dynamics of the model is described by a Markov process $r(t)$ having one component $r(t) = i(t)$. The process $r(t)$ takes values in the finite set of states $S = \{0, 1, \dots, k\}$ defined in accordance with link capacity. Let us by $P(i)$ denote the unnormalised values of stationary probabilities of $r(t)$. After normalisation, the value $P(i)$ denotes the mean portion of time when exactly i connections are established or i bandwidth units are occupied. From the system of state equations follows:

$$P(i) = \frac{\lambda}{i}\, P(i-1), \quad i = 1, 2, \dots, k\,. \tag{1}$$

Together with normalising condition the relations (1) gives the famous Erlang–B loss formula for estimation of the mean proportion of time

*COM Center, Building 345v, Technical University of Denmark, DK-2800 Kgs. Lyngby, Denmark, fax: (+45) 4593 6581, Email: vbi@com.dtu.dk

†Moscow City Telephone Company, 103804, Moscow, Petrovskii Bul. 12, Stroenie 3, Russia, fax: (095) 209 7854, Emails: {kostrov, stepanov}@rdc.ru

when all k bandwidth units are occupied:

$$E_k(\lambda) = \frac{\frac{\lambda^k}{k!}}{1+\lambda+\frac{\lambda^2}{2!}+\cdots+\frac{\lambda^k}{k!}}\,. \tag{2}$$

For dimensioning purposes (2) can be written in the form of recurrence:

$$E_k(\lambda) = \frac{\frac{\lambda}{k}\cdot E_{k-1}(\lambda)}{1+\frac{\lambda}{k}\cdot E_{k-1}(\lambda)}, \tag{3}$$

$$k = 1, 2, \ldots, \quad E_0(\lambda) = 1$$

The relation (3) is obtained by dividing upper and lower parts of (2) by $1+\lambda+\frac{\lambda^2}{2!}+\cdots+\frac{\lambda^{k-1}}{(k-1)!}$ and using the definition of the Erlang–B formula. The recursion (3) works with normalised values $E_k(\lambda)$ so its implementation does not suffer from numerical problems such as overflow or loss of precision that very often occur for large values of k. It is especially important to avoid such kinds of problems for future telecommunication networks where optical fibres provide great increase in number of channels that can be used for transmission.

In this paper we describe an alternative scheme for proving (3). This approach is quite simple and can easily be generalized to a number of multi-rate models. The general form of recurrence will be presented in Section 2 and a number of particular cases will be considered in Section 3.

2 General case

Let us consider a model of a service system with k service units. Let $p(i)$ be the mean proportion of time when exactly i service units are occupied. The main stationary performance measures of practical interest can be expressed as a function of $p(i)$, $i = 0, 1, \ldots, k$. Let us by a parameter B denote a measure used for dimensioning. A traditional dimensioning problem related with system planning is formulated as follows. For given input flow, find a value of k satisfying the inequality:

$$B \le \pi\,, \tag{4}$$

where π is a prescribed level of system functioning. Typically it will be the proportion of calls lost or the proportion of time the system is blocked for call servicing.

Usually only a limited number of $p(i)$'s are used in estimation of B and the number of $p(i)$'s used does not depend on the number of service units. For example, if each call of the input flow needs b units for its servicing then the proportion of time the system is blocked for call servicing can be estimated by the expression:

$$B = p(k-b+1) + p(k-b+2) + \cdots + p(k)\,.$$

Later we suppose that estimation of B for the traffic using k service units can be done by means of probabilities $p(k-b+1),\ p(k-b+2), \ldots, p(k)$, where b is some integer number independent of k. Solving the problem formulated by inequality (4) usually is going on according to the following recursive scheme. Let us by $P(i)$ denote unnormalised values of $p(i)$, $i = 0, 1, \ldots, k$, and suppose that for $i = 1, 2, \ldots, k$ the following relation is known for estimation of $P(i)$:

$$P(i) = f(P(i-1), P(i-2), \ldots, P(i-m)) \tag{5}$$

In (5) m is an integer, $P(0) = a$ and $P(i) = 0$ if $i < 0$. We suppose that its value is independent of the number of service units k. In some cases m is just equal to the maximum number of service units needed for call servicing. The function $f(\cdot)$ depends on the parameters of the model. Specific types of the function $f(\cdot)$ will be presented in the next section where we will consider examples of teletraffic models. The parameter a is any positive real number chosen as starting value of recurrence. Using (5) sequentially we can express all $P(i)$, $i = 1, 2, \ldots, k$, by $P(0)$. Then we find the normalising constant $Q = \sum_{i=0}^{k} P(i)$ and the normalised values of state probabilities:

$$p(i) = \frac{P(i)}{Q}, \quad i = 0, 1, \ldots, k\,. \tag{6}$$

We suppose that for any choice of a we obtain the same set of probabilities $p(i)$, $i = 0, 1, \ldots, k$, after normalisation. Next we calculate the performance measure B used in criteria (4). If number of service units k does not satisfy (4), then the value of k is increased and we repeat the described actions once more.

The implementation of this procedure has at least two negative aspects. They are clearly observed for a large number of service units k. For practically cases of interest almost all probability mass is located in states with large number of occupied service units. It means that states with small number of occupied service units have very small probabilities of existence. The state with number of occupied service units equals to zero has the smallest probability different from zero. So when we start to express all $P(i)$ through $P(0)$ we very quickly experience problems of overflow. Another negative aspect is the increase of computational efforts. To solve the formulated dimensioning problem we need to perform a run of (5) for each value of k serving as a candidate for desired solution. Each time we need to start calculation from $i = 0$.

Let us formulate a condition on implementation of function $f(\cdot)$ where we can solve this problem in one run as was done by recurrence (3) for service systems whose performance measures numerically are estimated by means of Erlang's B-formula. First let us indicate when necessary the number of available service units by using lower index for the corresponding set of probabilities. Let us suppose that for any $k > 1$ the implementation of recurrence (5) for number of service units $k-1$ and k and the same starting value gives sets of unnormalised probabilities $P_{k-1}(0) = a$, $P_{k-1}(1)$, $\ldots$, $P_{k-1}(k-1)$, and $P_k(0) = a$, $P_k(1)$, $\ldots$, $P_k(k-1)$, $P_k(k)$ satisfying the following relations:

$$P_k(i) = P_{k-1}(i)\,, \quad i = 0, 1, \ldots, k-1. \tag{7}$$

Property (7) allows us to rewrite recurrence (5) in a form dealing with normalised probabilities. Let us suppose that we know the normalised set of probabilities $p_{k-1}(0)$, $p_{k-1}(1)$,$\ldots$, $p_{k-1}(k-1)$ for number of service units $k-1$. Let us suppose that the number of service units is increased by one to k. Then taking as starting value $P_k(0) = p_{k-1}(0)$ and applying (5) we obtain according to property (7) the following set of unnormalised probabilities:

$$P_k(0) = p_{k-1}(0), \tag{8}$$

$$\dots \qquad \dots$$

$$P_k(k-1) = p_{k-1}(k-1),$$

$$P_k(k) = f(p_{k-1}(k-1), \dots, p_{k-1}(k-m)).$$

For normalised probability $P_k(k)$ we have:

$$p_k(k) = \frac{P_k(k)}{P_k(0) + \cdots + P_k(k)} \tag{9}$$

$$= \frac{P_k(k)}{p_{k-1}(0) + \cdots + p_{k-1}(k-1) + P_k(k)}$$

$$= \frac{P_k(k)}{1 + P_k(k)}.$$

Using (8) we rewrite (9) in the form:

$$p_k(k) \tag{10}$$

$$= \frac{f(p_{k-1}(k-1), \dots, p_{k-1}(k-m))}{1 + f(p_{k-1}(k-1), \dots, p_{k-1}(k-m))}.$$

At this step we need to calculate the performance measure B used for solving the dimensioning problem. Because the number of service units is increased by one we need to re-normalise probabilities $P_k(k-b+1) = p_{k-1}(k-b+1)$, $P_k(k-b+2) = p_{k-1}(k-b+2)$, ..., $P_k(k-1) = p_{k-1}(k-1)$, used for estimation of B (the value $p_k(k)$ has already been obtained by relation (10)). We can do this by means of equalities:

$$p_k(i) \tag{11}$$

$$= \frac{p_{k-1}(i)}{1 + f(p_{k-1}(k-1), \dots, p_{k-1}(k-b+1))},$$

$$i = k-b+1,\ k-b+2, \dots, k-1.$$

If the obtained value of B satisfies criteria (4), then the desired number of service units will be k. Other ways it is necessary to increase the number of service units k by one and to make a new estimation of B for a number of service units equal to $k+1$. To perform this step we need to have normalised values of $P_k(k-1) = p_{k-1}(k-1)$, $P_k(k-2) = p_{k-1}(k-2)$, ..., $P_k(k-m) = p_{k-1}(k-m)$ in order to use relation (10) in right way. We can do this by means of equality:

$$p_k(i) \tag{12}$$

$$= \frac{p_{k-1}(i)}{1 + f(p_{k-1}(k-1), \dots, p_{k-1}(k-m))},$$

$$i = k-m, k-m+1, \dots, k-1.$$

It is clear that implementation of (11) and (12) should be done once in the form:

$$p_k(i) \tag{13}$$

$$= \frac{p_{k-1}(i)}{1 + f(p_{k-1}(k-1), \dots, p_{k-1}(k-j))},$$

$$i = k-j,\ k-j+1, \dots\ k-1,$$

where $j = \max(b-1, m)$.

Computational efforts for current number of service units k is estimated by the total number of operations N_f that are needed for estimation of function $f(\cdot)$ (once) with the number of operations N_B that are needed for estimation of performance measure B (once) and $j+1$ operations of re-normalisation. So computational efforts to reach number of service units k starting from 1 is estimated by $O\{(N_f + N_B + j + 1)\, k\}$. Storage requirements are estimated by $O\{j+1\}$.

As result we have elaborated an approach that can be used for constructing numerically stable versions of recursive algorithms designed for estimation of performance measures of different teletraffic models. The results obtained are based on ideas from [2],[3]. If we compare the new algorithm with other stable algorithms known in this field [4],[5] we can conclude that the approach presented gives the result after very simple and obvious considerations. The approach can easily be generalised to other models. It has the smallest storage and computation requirement among known stable algorithms. Another good feature of the suggested approach is the possibility of increasing the efficiency by truncation of the state space used. Some aspects of implementation of this idea are considered in [2].

Of practical interest is the problem of finding stable recurrence for a number of multi-rate models that appear in the description of common sharing of bandwidth by a number of connections. A few results obtained in this direction will be presented in the next section, where we first look at single-rate models.

3 Particular cases

3.1 Family of teletraffic models described by finite birth and death processes

Let us consider a system having k service units. The time the model is staying in the state with i occupied service units is exponentially distributed with parameter $\lambda_i + \mu_i$. When this time expires, then with probability $\lambda_i/(\lambda_i + \mu_i)$ the number of occupied service units is increased by one (birth) and the system moves into state $i+1$. With probability $\mu_i/(\lambda_i + \mu_i)$ the number of occupied service units is decreased by one (death) and the system moves into state $i-1$. Adding necessary changes in boundary states ($i = 0$ and $i = k$) we obtain the family of teletraffic models described by finite birth and death processes. Among such models are finite source models and other widely used teletraffic models.

Let us by $p(i)$ denote the proportion of time when exactly i service units are occupied. Using the system of state equations it is easy to prove the relation

$$P(i) = \frac{\lambda_{i-1}}{\mu_i} \cdot P(i-1), \tag{14}$$

$$i = 1, 2, \ldots, k, \quad P(0) = 1,$$

which gives a specific type of the function $f(\cdot)$ (5) for models described by finite birth and death processes. Relation (10) for this case has the form:

$$p_k(k) = \frac{\frac{\lambda_{k-1}}{\mu_k} \cdot p_{k-1}(k-1)}{1 + \frac{\lambda_{k-1}}{\mu_k} \cdot p_{k-1}(k-1)}. \tag{15}$$

This equality gives the classical recurrence for Erlang's B-formula (3) for $\lambda_{k-1} = \lambda$, $\mu_k = k$.

So we find the general framework for constructing numerically stable recurrence formulæ for solving problems of dimensioning in one run for the family of teletraffic models described by finite birth and death processes. Storage requirements are $O\{1\}$ and computational efforts are estimated by $O\{k\}$, if the performance measure chosen is $p(k)$, the proportion of time the system spends in the state when all k accessible service units are occupied.

3.2 Multi-rate Poissonian models

Let us consider a single link traffic model, where the link transmission capacity is represented by k basic bandwidth units, and let us suppose that we have n incoming Poisson flows of calls with intensities λ_s, $s = 1, \ldots, n$. A call of s'th flow uses b_s bandwidth units for the time of connection. Without loss of generality we shall assume that the holding times all are exponentially distributed with the same mean value chosen to one, but it is known that the model considered is insensitive to the distribution of the holding time, and each flow may furthermore have individual mean holding times.

Let $i_s(t)$ denote the number of calls of the s'th flow served at time t. The model is described by an n-dimensional Markovian process of the type $r(t) = \{i_1(t), i_2(t), \ldots, i_n(t)\}$ with state space S consisting of vectors $(i_1, \ldots, i_n)$, where i_s is the number of calls of the s'th flow being served by the link under stationary conditions. The state space S is defined as follows: $(i_1, \ldots, i_n) \in S$, $i_s \geq 0$, $s = 1, \ldots, n$, $\sum_{s=1}^{n} i_s b_s \leq k$. Let us by $P(i_1, \ldots, i_n)$ denote the unnormalised values of stationary probabilities of $r(t)$. After normalisation the value $p(i_1, \ldots, i_n)$ denotes the mean proportion of time when exactly $\{i_1, \ldots, i_n\}$ connections are established. Assume that for state $(i_1, \ldots, i_n)$ the value i denotes the total number of occupied bandwidth units $i = i_1 b_1 + \cdots + i_n b_n$. The process of transmission of s'th flow is described by blocking probability π_s, $s = 1, \ldots, n$. Their formal definition through values of state probabilities are as follows (here and further, summations are for all states $(i_1, \ldots, i_n) \in S$ satisfying formulated condition, and by small characters we denote the normalised values of state probabilities):

$$\pi_s = \sum_{i+b_s>k} p(i_1, \ldots, i_n). \tag{16}$$

There are many algorithms for estimation of π_s. All of them are based on product form relations valid for $P(i_1, \ldots, i_n)$:

$$P(i_1, \ldots, i_n)$$
$$= P(0, \ldots, 0) \cdot \frac{\lambda_1^{i_1}}{i_1!} \cdot \frac{\lambda_2^{i_2}}{i_2!} \cdot \ldots \cdot \frac{\lambda_n^{i_n}}{i_n!},$$
$$(i_1, \ldots, i_n) \in S.$$

The most efficient calculation scheme for the model introduced is the recurrence algorithm first obtained in [6] and later also derived in [7],[8]. This algorithm exploits the fact that the performance measures (16) can be found if we for process $r(t)$ know probabilities $p(i)$ of being in the state where exactly i bandwidth units are occupied:

$$p(i) = \sum_{i_1 b_1 + \cdots + i_n b_n = i} p(i_1, \ldots, i_n)\,.$$

The corresponding formulas are as follows:

$$\pi_s = \sum_{i=k-b_s+1}^{k} p(i), \quad s = 1, 2, \ldots, n. \tag{17}$$

The unnormalised values of $P(i)$ are found by the recurrence:

$$P(i) = \frac{1}{i} \sum_{s=1}^{n} \lambda_s \, b_s \, P(i-b_s) \, I(i-b_s \geq 0), \quad (18)$$

$$i = 1, 2, \ldots, k,$$

where we usually let $P(0) = 1$ and function $I(\cdot)$ equals one if the formulated condition is fulfilled and otherwise equals zero. This relation gives specific type of function $f(\cdot)$ defined by (5) for the model. Recurrence (10) in this case has the form:

$$p_k(k) \quad (19)$$

$$= \frac{\frac{1}{k} \sum\limits_{s=1}^{n} \lambda_s \, b_s \, p_{k-1}(k-b_s) \, I(k - b_s \geq 0)}{1 + \frac{1}{k} \sum\limits_{s=1}^{n} \lambda_s \, b_s \, p_{k-1}(k-b_s) \, I(k - b_s \geq 0)}.$$

A one–run algorithm that at each step gives the normalised values of state probabilities which are necessary for solving the problem of dimensioning based on (17) is obtained by means of (19) and looks as follows:

Step 1. Put $p_0(0) = 1$.

Step 2. For fixed $k = 1, 2, \ldots,$ find normalised value of $p_k(k)$ by using relation (19)

$$p_k(k) \quad (20)$$

$$= \frac{\frac{1}{k} \sum\limits_{s=1}^{n} \lambda_s \, b_s \, p_{k-1}(k-b_s) \, I(k - b_s \geq 0)}{1 + \frac{1}{k} \sum\limits_{s=1}^{n} \lambda_s \, b_s \, p_{k-1}(k-b_s) \, I(k - b_s \geq 0)}.$$

Step 3. Let $b = \max\limits_{0 \leq s \leq n} (b_s)$. To calculate the normalised values of performance measures defined by (17) for number of bandwidth units equal to k and keep the possibility of performing the next step of recurrence we need the renormalised values of $p_{k-1}(i) \, I(i \geq 0)$, where $i = k-b+1, k-b+2, \ldots, k-1$ which were found when number of bandwidth units was $k-1$. We do this by using (13):

$$p_k(i) \quad (21)$$

$$= \frac{p_{k-1}(i) \, I(i \geq 0)}{1 + \frac{1}{k} \sum\limits_{s=1}^{n} \lambda_s \, b_s \, p_{k-1}(k - b_s) \, I(k - b_s \geq 0)},$$

$$i = k - b + 1, k - b + 2, \ldots, k - 1.$$

Step 4. Here we calculate the performance measures defined by (17), check the dimensioning criteria and either stop or continue the process of estimating the number of service units needed. When implementing the constructed version of recurrence algorithm we need to keep a vector of size $O\{b\}$ in computer memory. Computational efforts are estimated by $O\{(n + b)k\}$.

3.3 Multi-rate state-dependent models

Let us consider a single link traffic model, where the link transmission capacity is represented by k basic bandwidth units, and let us suppose that we have n incoming flows of calls. For s'th flow $\{s = 1, \ldots, n\}$, the time interval between successive call arrivals has a state-dependent exponential distribution with parameter $\lambda_s(i_s) = \alpha_s + i_s\beta_s$, where i_s is the number of calls of the s'th flow served by the link. A call of s'th flow uses b_s bandwidth units for the time of connection. Without loss of generality we shall assume that the holding times all are exponentially distributed with the same mean value chosen to one, but it is known that the model considered is insensitive to the distribution of the service time, and that each flow may have individual mean holding times.

Let us by $i_s(t)$ denote the number of calls of the s'th flow being served at time t. The model is described by an n-dimensional Markovian process of type $r(t) = \{i_1(t), i_2(t), \ldots, i_n(t)\}$ with state space S consisting of vectors $(i_1, \ldots, i_n)$, where i_s is the number of calls of the s'th flow served by the link under stationary conditions which impose some restrictions on values of α_s and β_s. The state space S is defined as follows:

$$(i_1, \ldots, i_n) \in S,$$

$$i_s \geq 0,\ s = 1, \ldots, n,\ \sum_{s=1}^{n} i_s b_s \leq k.$$

Let us denote by $P(i_1, \ldots, i_n)$ the unnormalised values of stationary probabilities of $r(t)$. After normalisation the value $p(i_1, \ldots, i_n)$ denotes the proportion of time when exactly $(i_1, \ldots, i_n)$ connections are established. Assume that for state $(i_1, \ldots, i_n)$ the value i denotes the total number of occupied bandwidth units $i = i_1 b_1 + \cdots + i_n b_n$. The process of transmission of the s'th flow is described by time blocking probability π_s and by call blocking probability ω_s, $s = 1, \ldots, n$. Their formal definition through values of state probabilities are as follows (here and further summations are for all states $(i_1, \ldots, i_n) \in S$ satisfying formulated condition and by small characters we denote the normalised values of state probabilities):

$$\pi_s = \sum_{i+b_s>k} p(i_1, \ldots, i_n)\,, \tag{22}$$

$$\omega_s = \frac{\sum\limits_{i+b_s>k} \lambda_s(i_s)\, P(i_1, \ldots, i_n)}{\sum\limits_{(i_1,\ldots,i_n)\in S} \lambda_s(i_s)\, P(i_1, \ldots, i_n)}\,.$$

This model like the previous one has product form representation for $P(i_1, \ldots, i_n)$:

$$P(i_1, \ldots, i_n) = P(0, \ldots, 0) \tag{23}$$

$$\times \frac{\prod\limits_{m=0}^{i_1-1} (\alpha_1 + m\beta_1)}{i_1!} \cdot \;\ldots\; \cdot \frac{\prod\limits_{m=0}^{i_n-1} (\alpha_n + m\beta_n)}{i_n!}\,.$$

Based on the property (23) a number of algorithms are constructed for estimating the performance measures (22). Let us formulate an algorithm established on results from [9]. This algorithm exploits the fact that the performance measures (22) can be found if we for the process $r(t)$ know auxiliary characteristics defined for the state where exactly i bandwidth units are occupied:

$$\begin{aligned} p(i) &= \sum_{i_1 b_1 + \cdots + i_n b_n = i} p(i_1, \ldots, i_n)\,, \\ m_s(i) &= \sum_{i_1 b_1 + \cdots + i_n b_n = i} i_s \cdot p(i_1, \ldots, i_n)\,. \end{aligned}$$

The corresponding formulas are as follows:

$$\pi_s = \sum_{i=k-b_s+1}^{k} p(i), \quad s = 1, 2, \ldots, n\,, \tag{24}$$

$$\omega_s = \frac{\sum\limits_{i=k-b_s+1}^{k} \{\alpha_s p(i) + \beta_s m_s(i)\}}{\sum\limits_{i=0}^{k} \{\alpha_s p(i) + \beta_s m_s(i)\}}\,. \tag{25}$$

Let us introduce auxiliary characteristic:

$$m_{s,k} = \sum_{i=0}^{k} \beta_s\, m_s(i)\,.$$

The unnormalised values of $P(i)$ and $M_s(i)$, $s = 1, 2, \ldots, n$ are found by recurrence:

$$P(i) = \frac{1}{i}\sum_{s=1}^{n} b_s\, M_s(i), \tag{26}$$

$$i = 1, 2, \ldots, k,$$

where

$$M_s(i) = I(i - b_s \geq 0)$$

$$\times\{\alpha_s P(i - b_s) + \beta_s M_s(i - b_s)\},$$

with starting value $P(0) = 1$ and function $I(\cdot)$ defined in the same way as for (18). This relation gives the type of function $f(\cdot)$ defined by (5) for studied model. Let us indicate when necessary for $M_s(i)$ or $m_s(i)$ the number of available service units k by using lower index k. For example, $M_{s,k}(i)$ or $m_{s,k}(i)$. Let us define $\mu_s(k)$ as

$$\mu_s(k) = I(k - b_s \geq 0)$$

$$\times\{\alpha_s\, b_s\, p_{k-1}(k - b_s) + \beta_s\, b_s\, m_{s,k-1}(k - b_s)\}$$

Recurrence (10) in this case has a form:

$$p_k(k) = \frac{\frac{1}{k}\sum\limits_{s=1}^{n}\mu_s(k)}{1 + \frac{1}{k}\sum\limits_{s=1}^{n}\mu_s(k)}. \tag{27}$$

A one-run algorithm that gives at each step the normalised values of the model's characteristics that are necessary for solving the dimensioning problem based on performance measures (24) is obtained by means of (27) and looks as follows.

Step 1. Let $p_0(0) = 1$.

Step 2. For fixed $k = 1, \ldots,$ find normalised value of $p_k(k)$ by using relation (27):

$$p_k(k) = \frac{\frac{1}{k}\sum\limits_{s=1}^{n}\mu_s(k)}{1 + \frac{1}{k}\sum\limits_{s=1}^{n}\mu_s(k)}. \tag{28}$$

Step 3. Let $b = \max\limits_{0\leq s\leq n}(b_s)$. To calculate the normalised values of performance measures defined by (24) for number of bandwidth units equal to k and to have the possibility of performing the next step of recurrence we need the renormalised values of $p_{k-1}(i)\, I(i \geq 0)$, where $i = k{-}b{+}1, k{-}b{+}2, \ldots, k{-}1$ and $m_{s,k-1}(i)\, I(i \geq 0)$, where $i = k{-}b{+}1, k{-}b{+}2, \ldots, k$, which were found when number of bandwidth units was $k{-}1$. We do this by taking (13) into account. For the same reason we need to re-normalise the characteristic $m_{s,k-1}$ which we need for calculation of the normalised value of ω_s. We have the following relations:

$$p_k(i) = \frac{p_{k-1}(i)\, I(i \geq 0)}{1 + \frac{1}{k}\sum\limits_{s=1}^{n}\mu_s(k)}, \tag{29}$$

$$i = k - b + 1, k - b + 2, \ldots, k - 1,$$

$$m_{s,k}(i) = \frac{m_{s,k-1}(i)\, I(i \geq 0)}{1 + \frac{1}{k}\sum\limits_{s=1}^{n}\mu_s(k)},$$

$$s = 1, 2, \ldots, n; \quad i = k - b + 1, k - b + 2, \ldots, k,$$

$$m_{s,k} = \frac{m_{s,k-1}}{1 + \frac{1}{k}\sum\limits_{s=1}^{n}\mu_s(k)},$$

$$s = 1, \ldots, n.$$

Step 4. Here we calculate the performance measure defined by (24), check the dimensioning criteria and either stop or continue the process of estimating the desired number of service units.

When implementing the constructed version of recurrence algorithm we need to keep a vector of size $O\{b\}$ in computer memory with computational efforts estimated by $O\{(n+b)\,k\}$ if we only use π_s for solving dimensioning problem, and we need to keep a vector of size $O\{n\,b\}$ in computer memory with computational efforts estimated by $O\{n\,b\,k\}$ if we use π_s and ω_s.

4 Conclusions

In this paper we describe the general framework for constructing numerically stable versions of recursive algorithms designed for estimation of performance measures of different teletraffic models. The results obtained are based on ideas from [2],[3] and generalise for multi-rate models the well known recurrence $E_k(\lambda) = \frac{\lambda}{k}\,E_{k-1}(\lambda)/\{1 + \frac{\lambda}{k}\,E_{k-1}(\lambda)\}\,, \quad k = 1, 2, \ldots,\ E_0(\lambda) = 1$, where $E_k(\lambda)$ is the Erlang–B formula for k servers with Poissonian input of intensity λ. The fact that we at each step of recurrence are dealing with normalised values of state probabilities used for estimation of main stationary performance measures is very well suited for solving problems of dimensioning. In addition to being numerically stable, the main positive features of the suggested approach are its simplicity and small storage requirements. Another good feature is the possibility to increase the efficiency of recursive algorithm by truncation of the state space used. Some aspects of realization of this idea are considered in [2].

References

[1] Hui, J. Y.: *Resource Allocation for Broadband Networks.* IEEE Journal on Selected Areas in Communications, Vol. 6 (1988) : 9, 1598–1608.

[2] Stepanov, S.N. & Iversen, V.B. & Kostrov, V.O.: *Optimised Dimensioning of Large Bandwidth Resources Under MPLS Technology.* Proc. of St. Petersburg Regional International Teletraffic Seminar on Telecommunication Network and Teletraffic Theory. St. Petersburg, Russia. January 2002, pp. 50–63.

[3] Iversen, V.B.: *Teletraffic Engineering. Chapter 11: Multi–dimensional Loss Systems.* COM Center, Technical University of Denmark. 2002. 338 pp.

[4] Nilsson, A. & Perry, M. & Gersht, A. & Iversen, V.B.: *On Multi–Rate Erlang-B Computations.* ITC 16, Sixteenth International Teletraffic Congress, Edinburgh, UK, June 1999. Elsevier 1999. pp. 1051–1060.

[5] Berezner, S.A. & Krzesinski, A.E.: *An Efficient Stable Recursion to Compute Multiservice Blocking Probabilities.* Performance Evaluation. Vol. 43 (2001) : 151—164.

[6] Fortet, R. & Grandjean, Ch.: *Congestion in a Loss System when Some Calls want Several Devices Simultaneously*. Electrical Communications, Vol. 39 (1964) : 4, 513–526. Paper presented at ITC–4, Fourth International Teletraffic Congress, London, UK, 15–21 July 1964.

[7] Kaufman, J.S.: *Blocking in a Shared Resource Environment*. IEEE Transactions on Communications, Vol. COM–29 (1981) : 10, 1474–1481.

[8] Roberts, J.W.: *A Service System with Heterogeneous User Requirements – Applications to Multi–Service Telecommunication Systems*. Pages 423–431 in Performance of Data Communication Systems and their Applications. G. Pujolle (editor), North–Holland Publ. Co. 1981.

[9] Delbrouck, L.E.N.: *On the Steady-State Distribution in a Service Facility carrying Mixtures of Traffic with Different Peakedness Factor and Capacity Requirements*. IEEE Transactions on Communications, Vol. COM–31 (1983) : 11, 1209–1211.