6. ON THE RATIONAL DETERMINATION OF THE NUMBER OF CIRCUITS

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1. Systems With Barred Access.

One might, perhaps, think that, for the determination of the number of circuits required, x, for a certain amount of traffic, y, or *vice versa*, it would be necessary to know only the formula of the probability of barred access:

$$B = \frac{\dfrac{y^x}{x!}}{1 + y + \dfrac{y^2}{2!} + \ldots . \dfrac{y^x}{x!}}, \tag{1}$$

combined with accordance in opinion as to the fixing of a suitable value of B; in that case, a table of the co-ordinate values of x and y would be sufficient. There are, however, the following points to be considered: in the first place, B must be multiplied by y (y designating the number of calls being originated during the unit of time, *i. e.* the average length of conversation), whereby the number of calls being barred during the unit of time is obtained; furthermore, it is not exactly $B \cdot y$ we want, but the reduction of $B \cdot y$ that will take place the moment we increase the number of available circuits by 1; this reduction we may call the "*Improvement,*" F_1 (in Danish: "*Forbedringen*"). When $B\ (x - 1)$ and $B\ (x)$ denote the values of B corresponding to $x - 1$ and x, the improvement caused by the transition from $x - 1$ circuits to x will be, then,

$$F_1 = y \cdot B\ (x - 1) - y \cdot B\ (x) \tag{2}$$

Table 1 gives F_1 for different values of x and y. (There are special reasons — which will be obvious later — for the method according to which the values of y have been selected). It will be noticed that F_1 decreases gradually as x increases. In order to find the particular value of x that is neither too great nor too small (for a certain value of y), it will be necessary to use a certain value of F_1 as our basis, and seek out in the table the place where one of two consecutive values of F_1 is slightly greater, and the other slightly smaller, than the basis value of F_1. In determining this basis value of F_1, allowance must be made for the inconvenience caused by preventing a call from getting through (this may be expressed in terms of money); also, for the costs per unit of time being incurred owing to the new circuit; the said costs comprise interest on the invested capital, depreciation, and maintenance of the circuit with accessories). If the employment of this, the most direct, method should give rise to any difficulties, it is of course possible to use another method; a single well-matched set of values of x and y could be taken as starting point; the values of F_1 for the said values of x and y could be found in the table; and the rest would pass off as described above. In all essentials, this chain of reasoning applies to the circuits interconnecting two exchanges as well as to the lines between the exchange and a subscriber with several lines.

Note: Instead of the above mentioned (exact) procedure, another (approximative) method may be used; the latter is especially useful and convenient for the greater values of x and y. The approximation improving with increasing values of x, we get:

$$F_1 = \frac{h \cdot \phi}{\phi_{-1}} + \frac{\phi^2}{\phi_{-1}^2}, \tag{3}$$

where

$$x = y + h\sqrt{y} \tag{4}$$

(here, ϕ and ϕ_{-1} denote simple and wellknown functions of h, *viz.* the Gaussian law-of-error function and its integral). The above shall not be proved here; but its correctness will appear rather clearly from the main table, 1, by traversing the latter along oblique lines issuing from the left-hand top corner. In consequence of the manner in which the values of y as contained in the table have been chosen, we have all along each line $x = y + h\sqrt{y}$, where h is constant. The values of F_1 are approximately constant along such a line, especially for the greater values of x, and consistent with formula (3).

A formula of the type

$$x = y + h\sqrt{y}$$

has already been employed for a long time in our Company to determine the number of junctions[1]), although without any absolutely satisfactory statement of reasons, as far as I know. The reasons can now be explained by means of the foregoing, also the proper significance of the constant h.

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[1]) This formula was published by *P. V. Christensen* in 1913 (see the foot-note p. 16).

Table 1. The improvement F_1. (Formula 2).

$x-y$	$y=1$	$y=4$	$y=9$	$y=16$	$y=25$	$y=36$	$y=49$	$y=64$	$y=81$	$y=100$
0	0.5000	0.5601	0.5837	0.5963	0.6040	0.6092	0.6131	0.6159	0.6181	0.6200
1	0.3000	0.4464	0.5072	0.5389	0.5582	0.5714	0.5807	0.5878	0.5933	0.5975
2	0.1375	0.3276	0.4242	0.4770	0.5094	0.5311	0.5465	0.5580	0.5671	0.5742
3	0.0471	0.2177	0.3397	0.4121	0.4577	0.4884	0.5104	0.5269	0.5394	0.5495
4	0.0123	0.1293	0.2581	0.3463	0.4045	0.4444	0.4731	0.4945	0.5111	0.5244
5	0.0026	0.0683	0.1856	0.2821	0.3506	0.3991	0.4345	0.4612	0.4819	0.4983
6	0.0004	0.0321	0.1252	0.2221	0.2977	0.3537	0.3953	0.4270	0.4517	0.4713
7	0.0001	0.0135	0.0793	0.1684	0.2471	0.3086	0.3558	0.3923	0.4210	0.4441
8		0.0051	0.0471	0.1229	0.2001	0.2650	0.3167	0.3575	0.3900	0.4162
9		0.0018	0.0263	0.0861	0.1579	0.2237	0.2785	0.3229	0.3588	0.3881
10		0.0006	0.0138	0.0579	0.1213	0.1854	0.2419	0.2891	0.3279	0.3599
11		0.0002	0.0067	0.0374	0.0906	0.1507	0.2071	0.2561	0.2974	0.3320
12		0.0001	0.0032	0.0231	0.0658	0.1201	0.1749	0.2246	0.2676	0.3043
13			0.0014	0.0138	0.0464	0.0938	0.1456	0.1948	0.2388	0.2769
14			0.0006	0.0079	0.0318	0.0718	0.1192	0.1671	0.2113	0.2506
15			0.0003	0.0044	0.0212	0.0538	0.0963	0.1417	0.1853	0.2251
16			0.0001	0.0023	0.0138	0.0394	0.0765	0.1186	0.1610	0.2008
17				0.0012	0.0087	0.0283	0.0698	0.0982	0.1386	0.1778
18				0.0006	0.0054	0.0199	0.0460	0.0802	0.1181	0.1561
19				0.0003	0.0032	0.0138	0.0348	0.0647	0.0997	0.1361
20				0.0001	0.0019	0.0093	0.0260	0.0616	0.0833	0.1177
21				0.0001	0.0011	0.0062	0.0190	0.0406	0.0689	0.1009
22					0.0006	0.0040	0.0137	0.0315	0.0564	0.0859
23					0.0003	0.0026	0.0097	0.0242	0.0457	0.0724
24					0.0002	0.0016	0.0068	0.0183	0.0367	0.0606
25					0.0001	0.0010	0.0048	0.0137	0.0291	0.0502
26						0.0006	0.0032	0.0101	0.0229	0.0413
27						0.0004	0.0021	0.0074	0.0178	0.0337
28						0.0002	0.0014	0.0053	0.0137	0.0272
29						0.0001	0.0009	0.0038	0.0104	0.0218
30						0.0001	0.0006	0.0027	0.0079	0.0173
31							0.0004	0.0019	0.0059	0.0136
32							0.0002	0.0013	0.0043	0.0107
33							0.0001	0.0009	0.0032	0.0083
34							0.0001	0.0006	0.0023	0.0063
35							0.0001	0.0004	0.0017	0.0048
36								0.0003	0.0012	0.0037
37								0.0002	0.0008	0.0027
38								0.0001	0.0006	0.0020
39								0.0001	0.0004	0.0015
40									0.0003	0.0011
41									0.0002	0.0008
42									0.0001	0.0006
43									0.0001	0.0004
44									0.0001	0.0003
45										0.0002

2. Systems With Delay.

Here, as in the preceding section, we must find out about the improvement obtained by adding 1 circuit to those already existing.

We may start from the average delay formula for x lines, supposing exponentially distributed holding time:

$$M(x) = \frac{1}{x-y} \cdot \frac{1}{D(x) - D(x-1)}, \tag{5}$$

where

$$D(x) = \frac{1}{B(x)} \quad \text{and} \quad D(x-1) = \frac{1}{B(x-1)}$$

and $B(x)$ and $B(x-1)$ have the same significance as above. Since, on the average, y calls are originated during the unit of time, we have to multiply by y, and so we get:

$$y \cdot M(x) = \frac{y}{x-y} \cdot \frac{1}{D(x) - D(x-1)},$$

which is the total average delay per unit of time, or, if you like, the mean number of simultaneously waiting calls. For $x-1$ we have, correspondingly,

$$y \cdot M(x-1) = \frac{y}{x-1-y} \cdot \frac{1}{D(x-1) - D(x-2)}.$$

Hence, by transition from $x-1$ to x circuits the improvement will be:

$$F_2 = y \cdot M(x-1) - y \cdot M(x) = \frac{y}{x-1-y} \cdot \frac{1}{D(x-1) - D(x-2)} - \frac{y}{x-y} \cdot \frac{1}{D(x) - D(x-1)}. \tag{6}$$

Table 2 is a table of the quantities F_2 thus determined. As was the case in the above, F_2 will be found to decrease as x increases; consequentially, there must be an optimum point where F_2 will about balance against the costs in connexion with adding one more circuit, or, expressed more exactly, the next improvement will be just too small to justify the costs of another new circuit. Although the word *circuit* has been used all along, it should be remembered that the delays may as well be due to the simultaneous engagement of each of a group of teamworking *operators*, instead of a group of cooperating circuits; and this leads to the question, How many operators should be set to work the traffic concerned? Mr. *K. Moe* has been working on this problem; as might be expected, his reasoning and procedure — although somewhat different in form — in reality agree with the above.

The results as stated in table 2 may be interpreted very clearly and plainly if we, in the following, rather stick to the last mentioned application of the theory. The whole problem consists of two points: the subscriber's waiting time, receiver in hand; and the operators' hours of service at the exchange (not exactly the time they are actually working their positions). It is necessary to have a certain basis value of F_2 in order to take full advantage of the table; now, it is obvious that if we put $F_2 = 1$, it means that an average subscriber's waiting time at his telephone is rated at the same value as the operator's hours of service at the exchange; $F_2 = \frac{1}{2}$ means that the subscriber's time is twice as valuable, &c.

Note: Here, too, another (approximative) method may be used instead of the (exact) procedure just mentioned, the former being especially useful and convenient for great values of x:

$$F_2 = \frac{\phi^2 (1 + h^2) + \phi_{-1} \phi (2h + h^3)}{(\phi + h\phi_{-1})^2 \cdot h^2}, \tag{7}$$

$$x = y + h\sqrt{y}. \tag{8}$$

The proof of this shall not be given here; but its correctness will appear rather clearly from the main table, 2, by transversing the latter along oblique lines issuing from the left-hand top corner. In consequence of the manner, in which the values of y as contained in the table have been chosen, we have $x = y + h\sqrt{y}$ all along such a line. It will be noticed that the values of F_2 are approximately constant along such a line, especially for the greater values of x, and consistent with formula (7).

Table 2. The improvement F_2. (Formula 6).

$x-y$	$y=1$	$y=4$	$y=9$	$y=16$	$y=25$	$y=36$	$y=49$	$y=64$	$y=81$	$y=100$
2	0.288	1.647	4.081	7.545	12.025	17.505	23.995	31.487	39.961	49.465
3	0.039	0.389	1.139	2.257	3.726	5.537	7.689	10.181	13.007	16.168
4	0.006	0.121	0.443	0.968	1.680	2.570	3.634	4.864	6.272	7.845
5	0.001	0.040	0.194	0.477	0.907	1.400	2.026	2.760	3.596	4.532
6		0.013	0.088	0.251	0.500	0.830	1.236	1.715	2.263	2.884
7		0.004	0.040	0.136	0.296	0.517	0.795	1.128	1.514	1.951
8		0.001	0.018	0.074	0.176	0.331	0.530	0.771	1.053	1.374
9			0.008	0.040	0.109	0.216	0.360	0.539	0.751	0.997
10			0.003	0.022	0.067	0.142	0.248	0.384	0.548	0.737
11			0.001	0.012	0.040	0.094	0.172	0.276	0.403	0.554
12			0.001	0.006	0.024	0.062	0.120	0.200	0.301	0.421
13				0.003	0.015	0.040	0.084	0.145	0.225	0.321
14				0.002	0.009	0.026	0.058	0.106	0.169	0.248
15				0.001	0.005	0.017	0.040	0.077	0.127	0.191
16					0.003	0.011	0.028	0.056	0.096	0.148
17					0.002	0.007	0.019	0.040	0.072	0.114
18					0.001	0.004	0.013	0.029	0.053	0.088
19						0.003	0.009	0.021	0.040	0.068
20						0.002	0.006	0.015	0.030	0.052
21						0.001	0.004	0.010	0.022	0.040
22						0.001	0.003	0.007	0.016	0.031
23							0.002	0.005	0.012	0.024
24							0.001	0.004	0.009	0.018
25							0.001	0.002	0.006	0.014
26								0.002	0.005	0.010
27								0.001	0.003	0.008
28								0.001	0.002	0.006
29								0.001	0.002	0.004
30									0.001	0.003
31									0.001	0.002
32									0.001	0.002
33										0.001
34										0.001
35										0.001