10. AN ELEMENTARY THEORETICAL STUDY OF THE INDUCTION COIL IN A SUBSCRIBER'S TELEPHONE APPARATUS

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1. Introduction.

An ordinary telephone instrument of the Local Battery type contains, in addition to a telephone receiver and a microphone with appurtenant battery, a transformer (toroidal induction coil, or plain induction coil) the primary winding of which is part of the local or microphone circuit, while the secondary is connected in series with the receiver and the external conductor. Even if these items are taken for granted, the description of the instrument would not be complete without mention of quite a lot of quantities of mechanical, electrical, or magnetical nature. In practice, a good many of these can be chosen at discretion, alone with a view to the greatest possible efficiency of the instrument. The most important problem in a theoretical investigation is therefore the determination of such values of the constants as will give the best possible result; it should be remembered, however, that in each particular case a certain latitude is permissible, having no perceptible effects in practice, not even by systematic speech tests.

2. Formulating the Problem; Denotations.

In order to find out which type of transformer is the best, we must know, firstly, the resistance r_1 of the microphone, and secondly, the characteristic impedance of the conductor, *i. e.* the apparent impedance of the infinitely long line, and the impedance of the receiver; or rather, just the sum r_2 of the latter two, which is supposed to be a simple resistance. We assume that the two instruments under consideration are absolutely identical, and connected by a long conductor. The resistance variations of the microphone are supposed to be small; the radian frequency is called ω ($= 2\pi \cdot$ frequency). It is difficult — even after the latest microphone investigations — to account for what actually happens when the resistance connected in series with a microphone is increased or decreased (unless the battery voltage is changed simultaneously so that the current remains constant); but we can evade this difficulty by supposing that the resistance of the primary winding is insignificant as compared to r_1; the resistance of the secondary is, similarly, considered small in proposition to r_2. It is also assumed that eddy-currents and hysteresis are negligible quantities. The last mentioned conditions are easily satisfied in practice if the available space is not too restricted, especially if it is possible to use thin core wires made of a good quality of iron. Now, we want to determine the two coefficients of self-induction, l_1 and l_2, and the coefficient of mutual induction, m; we will suppose that

$$\frac{l_2}{m} = \frac{m}{l_1}, \tag{1}$$

i. e. there is no leakage flux.

3. *Fundamental Equations.*

Under the above suppositions we can find the resultant impedance of the transformer by disconnecting the microphone and measuring the primary, or by disconnecting the receiver and conductor and measuring the secondary. We get, respectively,

$$R_1 = \frac{l_1 r_2}{\frac{r_2}{i\omega} + l_2} \tag{2}$$

$$R_2 = \frac{l_2 r_1}{\frac{r_1}{i\omega} + l_1}, \tag{3}$$

using the mathematical operator i in the now familiar manner. Furthermore we can find the alternating current values in the following circuits: the microphone circuit of the transmitting instrument; the telephone receiver of the latter, or the near end of the conductor; the far end of the conductor, or the telephone receiver of the receiving instrument; the microphone circuit of the receiving instrument. The last of these alternating current values is of no importance at all, however; the first three will be, respectively,

$$I_0 = \frac{Ed\left(\frac{r_2}{i\omega} + l_2\right)}{r_1\left(r_1 l_2 + r_2 l_1 + \frac{r_1 r_2}{i\omega}\right)} \tag{4}$$

$$I_1 = \frac{Edm}{r_1\left(r_1 l_2 + r_2 l_1 + \frac{r_1 r_2}{i\omega}\right)} \tag{5}$$

$$I_2 = \frac{2\,EdZe^{-\gamma l}\,m\left(\frac{r_1}{i\omega} + l_1\right)}{r_1\left(r_1 l_2 + r_2 l_1 + \frac{r_1 r_2}{i\omega}\right)^2} \tag{6}$$

It has been necessary to introduce some new denotations here, *viz.*

E = the battery e. m. f. or voltage
d = half of the variation of the microphone resistance during speech
Z = the characteristic impedance of the conductor
z = the numerical value of Z
l = the length of the conductor
γ = the propagation constant of the conductor $= \beta + i\alpha$
β = the real number component of γ.

In the case of the present investigation, however, all these quantities are constants, so that we may put

$$\frac{2\,Edze^{-\beta l}}{r_1} = k \tag{7}$$

Hence, considering especially I_2 (and only its numerical value, or amplitude, $|I_2|$, as the phase is of no importance in this respect), we obtain

$$|I_2|^2 = k^2 \frac{m^2\left(\frac{r_1^2}{\omega^2} + l_1^2\right)}{\left((r_1 l_2 + r_2 l_1)^2 + \frac{r_1^2 r_2^2}{\omega^2}\right)^2} \tag{8}$$

or, because $l_1 l_2 = m^2$,

$$|I_2|^2 = k^2 \frac{m^2\left(\frac{r_1^2}{\omega^2} + l_1^2\right)}{\left(r_2^2 l_1^2 + \frac{r_1^2 m^4}{l_1^2} + 2\,r_1 r_2 m^2 + \frac{r_1^2 r_2^2}{\omega^2}\right)^2} \tag{9}$$

or, by substituting L for l_1^2,

$$|I_2|^2 = k^2 \frac{m^2\left(\frac{r_1^2}{\omega^2} + L\right)}{\left(r_2^2 L + \frac{r_1^2 m^4}{L} + 2\, r_1 r_2 m^2 + \frac{r_1^2 r_2^2}{\omega^2}\right)^2} \tag{10}$$

4. First Condition To Be Satisfied.

If we, temporarily, let L be a constant, we must now — according to (10) — aim at securing a minimum value of the quantity

$$\left(r_2^2 L + \frac{r_1^2 r_2^2}{\omega^2}\right)\frac{1}{m} + 2\, r_1 r_2 m + \frac{r_1^2}{L} m^3;$$

we have, then,

$$\frac{3\, r_1^2}{L} m^2 + 2\, r_1 r_2 - \left(r_2^2 L + \frac{r_1^2 r_2^2}{\omega^2}\right)\frac{1}{m^2} = 0, \tag{11}$$

or, by substituting M for m^2,

$$\frac{3\, r_1^2}{L} M^2 + 2\, r_1 r_2 M - \left(r_2^2 L + \frac{r_1^2 r_2^2}{\omega^2}\right) = 0 \tag{12}$$

$$\frac{M}{L} = \frac{r_2}{3\, r_1}\left(-1 + \sqrt{4 + \frac{3\, r_1^2}{L\omega^2}}\right) \tag{13}$$

The value thus found will always be real and positive.

5. Second Condition To Be Satisfied.

Now let us regard $\frac{M}{L} = x$ as a constant. We have, from (8):

$$|I_2|^2 = k^2 \frac{M\left(\frac{r_1^2}{\omega^2} + L\right)}{\left(L\left(r_1 \cdot \frac{M}{L} + r_2\right)^2 + \frac{r_1^2 r_2^2}{\omega^2}\right)^2} \tag{14}$$

$$|I_2|^2 = k^2 \frac{xL\left(\frac{r_1^2}{\omega^2} + L\right)}{\left(L\,(r_1 x + r_2)^2 + \frac{r_1^2 r_2^2}{\omega^2}\right)^2} \tag{15}$$

The maximum value of this is obtained for

$$L = \frac{r_1^2}{\omega^2 \left(\frac{(r_1 x + r_2)^2}{r_2^2} - 2 \right)}, \tag{16a}$$

provided that $r_1 x > r_2 (\sqrt{2} - 1)$; if, on the other hand, $r_1 x < r_2 (\sqrt{2} - 1)$, it is necessary that

$$L = \infty, \tag{16b}$$

since L cannot be negative.

6. Combined Result.

It now remains to combine the results obtained in the two preceding sections. No serviceable result can be derived from (13) and (16a) taken together. (13) together with (16b) give the following result,

$$L = \infty \tag{17}$$

$$x = \frac{r_2}{3 r_1} \tag{18}$$

Here, the ratio n of the number of turns in the primary and secondary windings can be introduced; this ratio being the square root of x, we have

$$n = \sqrt{\frac{r_2}{3 r_1}}. \tag{19}$$

The corresponding maximum value of $|I_2|$ will be

$$|I_2|_{\max} = \frac{3\sqrt{3}\, Edze^{-\beta l}}{8 r_1 r_2 \sqrt{r_1 r_2}}, \tag{20}$$

or

$$|I_2|_{\max} = \frac{3\sqrt{3}}{16} k \cdot \frac{1}{r_2 \sqrt{r_1 r_2}} \tag{21}$$

The corresponding values of R_1 and R_2 will be

$$R_1 = 3 r_1 \tag{22}$$

$$R_2 = \tfrac{1}{3} r_2 \tag{23}$$

7. *Comparison of Different Transformers.*

It is often useful to have an algebraic expression for the serviceableness of different transformers. Then, a comparison of each individual transformer to the best possible one suggests itself naturally; the ratio of the two values of $|I_2|$ concerned being denoted by f, we arrive at the formula,

$$f = \frac{16}{3\sqrt{3}} \cdot \frac{\sqrt{x\left(\frac{r_2^2}{l_1^2\omega^2} + \frac{r_2^2}{r_1^2}\right)\cdot\frac{r_2}{r_1}}}{\left(x + \frac{r_2}{r_1}\right)^2 + \frac{r_2^2}{l_1^2\omega^2}}, \tag{24}$$

or

$$f = \frac{16}{3\sqrt{3}} \cdot \frac{n\sqrt{\left(\frac{r_2^2}{l_1^2\omega^2} + \frac{r_2^2}{r_1^2}\right)\frac{r_2}{r_1}}}{\left(n^2 + \frac{r_2}{r_1}\right)^2 + \frac{r_2^2}{l_1^2\omega^2}}. \tag{25}$$

8. *Example.*

Putting $r_1 = 20$; $r_2 = 2000$; $\omega = 5000$, we get a rather typical example, *viz.*

$$n = \sqrt{\frac{100}{3}} = 5 \cdot 8 \tag{26}$$

and furthermore, by means of (25), the following "double entry" table of f as a function of l_1 and n:

n \ l_1	·000	·001	·002	·003	·004	·005	·006	·007	·008	·009	·010	∞
5	·000	·362	·619	·768	·850	·896	·923	·939	·950	·958	·963	·989
6	·000	·427	·706	·849	·917	·950	·968	·978	·984	·988	·990	·999
7	·000	·488	·775	·898	·947	·965	·972	·975	·976	·976	·976	·971
8	·000	·544	·824	·919	·944	·948	·945	·941	·937	·934	·931	·916
9	·000	·593	·852	·914	·917	·907	·895	·886	·879	·873	·869	·846
10	·000	·635	·861	·888	·871	·850	·833	·820	·810	·803	·797	·770
11	·000	·669	·853	·848	·814	·786	·764	·749	·738	·730	·723	·694
12	·000	·694	·830	·797	·752	·718	·694	·678	·666	·657	·651	·621
13	·000	·710	·797	·740	·688	·651	·627	·610	·598	·589	·583	·553
14	·000	·718	·756	·682	·625	·587	·563	·547	·535	·527	·520	·492
15	·000	·717	·709	·624	·565	·528	·504	·489	·478	·470	·464	·437

Such a table can be utilized in various ways. Thus, values taken along vertical or horizontal lines may be used for plotting curves; or l_1 and n may be regarded as right-angled coordinates for points in a plane, and so curves can be plotted in this plane corresponding to $f = 0{\cdot}90$, $0{\cdot}80$, &c. In practice, all transformers belonging on the proper side of the curve for $f = 0{\cdot}90$ will probably not be much inferior to the theoretically correct one. — The reasons why the curves mentioned are not shown here, are that they are so easy to plot, and that the table chiefly gives the same information.

9. *Articulation.*

Apart from $\omega = 5000$, other somewhat smaller and larger values must be considered, if not only a high power level, but also high intelligibility in the transmission of speech is desired. For this purpose, however, no special calculation is necessary; as formula (25) shows, a change of ω can always be regarded as equivalent to a change of l_1, and the desired values of f as corresponding to different values of ω are consequently to be found in a horizontal line of the above mentioned table (or from the correspondingly derived curve). Here, only the articulation reduction due to the transformer has been taken into consideration (and r_2 is accordingly regarded as being constant for the different frequencies); as it happens, this reduction is generally of no importance as compared to that due to the variation of the β of the cables in particular, and perhaps — to some extent — to the variation of the d of the microphone.

10. *Resistance of Transmitter and Receiver.*

The formula (20) gives, at least, some information as to the question of what resistance values should be chosen for the microphone and the telephone receiver, although it is an indication only — not a directly applicable rule — since neither of these is so simple a device as the transformer. Let the best telephone be the one that receives the most energy; we shall then find that the resistance of the telephone should be about $\frac{1}{3} r_2$, or about $\frac{1}{2} z$. Regarding the microphone we may work on the hypothesis that d and r_1 are proportionals; there seems to be reason, also, for assuming that the battery voltage and the square root of r_1 should be proportionals, *e. g.* with a view to the necessary cooling of the microphone. Under these suppositions, all values of r_1 may prove to be equally good.

11. Inductive Shunt.

It seems natural to ask whether some other device might not be just as useful, or even more so, than a transformer; it would rather have to be a low-resistance shunt with a high coefficient of self-induction. The transformer which, according to the above, was selected as the best would then, for $r_2 = 3\ r_1$, be equivalent to the shunt. In practice, however, we always have $r_2 > 3\ r_1$, and the transformer is therefore to be preferred.

12. Conclusion.

We have in the foregoing been dealing with the Local Battery instrument, under presuppositions chosen so as to make the calculations fairly simple, but without departing essentially from what is usual or possible in practice. The supposition that the impedance r_2 is a simple resistance may, of course, be abandoned, and an arbitrary impedance taken instead; but there is scarcely anything to be gained by that. In the essentials, the problem in connexion with Central Battery instruments is the same as above, although the presuppositions should, perhaps, be chosen somewhat differently; besides, the transformer is not quite so important here as in the case of the L. B. instrument.